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# Computer Algebra Systems in Teaching and Research

## Volume VIII

Siedlce University of Natural Sciences and Humanities
Institute of Mathematics and Physics

SIEDLCE 2019

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## Volume VIII

**Siedlce University of Natural Sciences and Humanities**

**Siedlce 2019**

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Cover designed by Paweł Trojanowski



ISBN 978-83-7051-956-8

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### I. Mathematical Modeling and Differential Equations



### II. Problems of Classical and Celestial Mechanics

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## Acceleration of the Second Order – Jerk of a Rigid Body Rotates around a Fixed Point*

*Paper dedicated to memory of my professor of Mechanics,*
*Professor Dr. Ing. Dipl. Math. Danilo P. Rašković (1910-1985)*
*and half century from first common publication*

**Katica R. (Stevanović) Hedrih***

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*"Acceleration of the second order (jerk or jark) of a material point that moves at a constant sectoral velocity."*

Another paper is titled:

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The first part of this second work was shown using the matrix notation for description of the kinematics of motion, and authored by my brilliant Professor dr. Ing. Dipl. Math. Danilo Rašković (1910-1985). He was one of the head of the Department of Mechanics at Mathematical Institute of the Serbian academy of sciences and arts, and author of 150.000 examples of different books in area of theoretical and applied mechanics, published in Serbian language, and used by numerous generations of students of technical sciences. The second part of this work is shown by the vector notation and represents the author's scientific results. And the entire work belongs to the classical field of the kinematics of the body rotation around a fixed point.

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# Acceleration of the second order – jerk of a rigid body rotates around a fixed point*

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Acceleration of the second order – jerk (Ruch, Rucken, Jerk, pulse)(Word forms: jerks, jerking, jerked) plays an increasing role in various domains of mechanics and physics. The aim of this paper is to determine this acceleration of the second order, or shorter *jerk,* when a rigid body rotates around a fixed point. The corresponding relations for thee instantaneous

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angular velocities, angular accelerations, velocities, acceleration of the second order and jerk are expressed in matrix form and referred to the fixed point and instantaneous axes. The Rivals theorem is enlarged on the jerk also, and the natural components of the acceleration of the second order - jerk vector of a rigid body rotation around a fixed point are given. The case of the regular progressive precession is treated, also.

In concluding remarks, the contents of the original and new results, will be displayed on the topic of the dynamics of rotating the rigid body around a fixed point using the vectors of mass moments that are related to the pole and axis, introduced by the author in 1992. From this area, in special cases of rigid body dynamics when rotating around a fixed point, the most famous works and solutions are: Euler's solution, Lagrange's solution and the solution of Sofia Kovaljevskya. The general solution of the dynamics of the rigid body differential equations when rotates around a fixed point has not been found to this date.

About twenty years ago, the lecturer attended the Conference in Donetsk, organized by the Institute of Mechanics NANU from Donetsk, one *Round Table* and a discussion on one book and one paper, by two independent researchers, who individually claimed to have found a general solution a system of differential equations of the dynamics of a rigid body around a fixed point. It was a real "scientific octave" between the authors and the opponent's important world scientist. The paper will make an attempt to display this discussion in Appendix of this paper..

**Key words:** Acceleration of the second order, jerk, rigid body that rotates around a fixed point, instantaneous axis, angular velocity, matrix method, vector method, mass moment vectors.

## 1. Introduction

If one point $A$ of the rigid body, see Figure 1. and Reference [1], whose representative is a rigid triangle $ABC$, is immobile, then the other two points $B$ and $C$ are moving along spherical surfaces, with centres in a fixed point $A$. If we assume that the radii of these spherical surfaces are equal, then both points $B$ and $C$ move along one spherical surface - one sphere. If through the points $A$, $B$ and $C$ we set one straight plane, which is a fixation at a point $A$, then the points $B$ and $C$ are also located on the main bigger circle of the sphere, upon which they move. Then, the arch $\hat{BC}$ repels the rod, which means that the spherical movement of the rod $\overline{BC}$ around the fixed point $A$ is realized. While the plane of the plane movement of the rigid body was transmitting the motion of one plane from the intersection with the rigid body, in this plane of the motion, here in the spherical motion of the rigid body, we observed the movement of the spherical triangle figure, the cross-section of the rigid body with a immobile spherical surface, on that surface. From this analogy we conclude, that the spherical motion of a rigid body around a fixed point has three degrees of freedom of movement, as well as a flat planar motion of a rigid body (see Reference [1] and Figure 1.).

Let us mention a very well-known theorem defined by French scientist d’Alambert-s (Jean Le Rond d’Alambert, 1717-1783, “Reserches sur la prècession des equinoxes”, Paris, 1749)  in following formulation:
*“A rigid body, whose one point is immobile, can be sung from one position to the other nearby position by rotating around the axis passing through the fixed point”.*

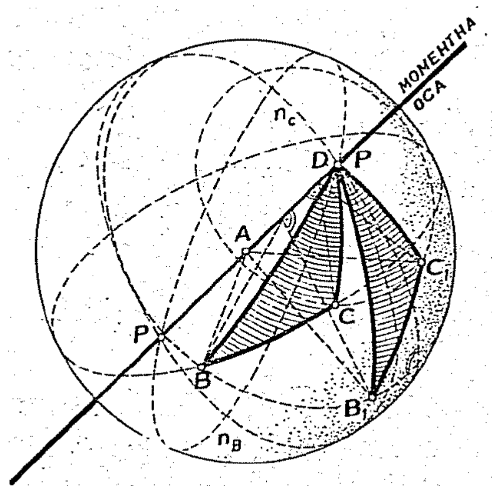



**Figure 1.** Rigid body rotation around a fixed point - Spherical motion of a rigid body arounfd fixed point (Figure from Reference [1])

If the rotation of a rigid body is made around an axis passing through a fixed point, for an infinite small angle, the axis is called the momentary-instantaneous - the rotation axis. The role that, in the plane motion, had the current instantaneous pole of rotation, in analogy, in spherical motion, has the instantaneous axis of rotation rigid body around fixed point. All of its points at that moment have no velocity.

Position of a free rigid body is defined with the basic rigid triangle, i.e. with 9 coordinates, so that it is rigid. It requires 6 coordinates, which are independent because there are three connections from the conditions of stiffness of the rigid body triangle. If the rigid body is fixed at one point, it has three links-constraints that prevent three translations, and the rigid body in the rotation around the fixed point has three degrees of freedom of movement.

Thus, even the spherical movement of a rigid body has a three degrees of freedom of movement. The plane motion of a rigid body is represented by the coplanar motion of a rigid triangle and the analogous, spherical motion of a rigid body is represented by the motion of a rigid spherical triangle over the sphere. This means that the plane movement of the triangle can be considered a special case of spherical motion, when the radius of the sphere is infinitely large. The angular velocity of a rigid body in spherical motion is a sliding vector that falls in the direction of the current momentary axis of rotation, as well as in plane motion. Now, by analogy, the velocity $\vec{v}$ of each point $B$ of the rigid body, which rotates around a fixed point $A$, is determined by Euler's equation in the form:

$$\vec{v} = \left[\Omega, \overline{AB}\right] \tag{1}$$

## 2. Matrix relations of the kinematics of the rotation of the rigid body around a fixed point

In the fixed point $O$ of a rigid body, we will adopt the coordinate start of a fixed coordinate system $Ox_1x_2x_3$ (Figure 2). When rotating, this triad-fixed coordinate system, will move to the position $Oy_1y_2y_3$. The movements of the moving coordinate system are determined in the directions of the axes of the fixed coordinate system of the nine angles, that is, with the direction of the cosine of the directions. This is the matrix of coordinates of the unit vectors of the matrix of rows $\left(\vec{j}_r\right) = \left( {}_r \quad {}_r \quad {}_r \right)$, $r = 1,2,3$ , the moving coordinate system compared to the base coordinate system. This matrix, as it is known, is orthogonal, and will be:

$$\mathbf{A} = \begin{pmatrix} \left(\vec{j}_1\right) \\ \left(\vec{j}_2\right) \\ \left(\vec{j}_3\right) \end{pmatrix} = \begin{pmatrix} {}_1 & {}_1 & {}_1 \\ {}_2 & {}_2 & {}_2 \\ {}_3 & {}_3 & {}_3 \end{pmatrix} \qquad \mathbf{A}' = \mathbf{A}^{-1} \qquad \mathbf{A}\mathbf{A}' = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \tag{2}$$

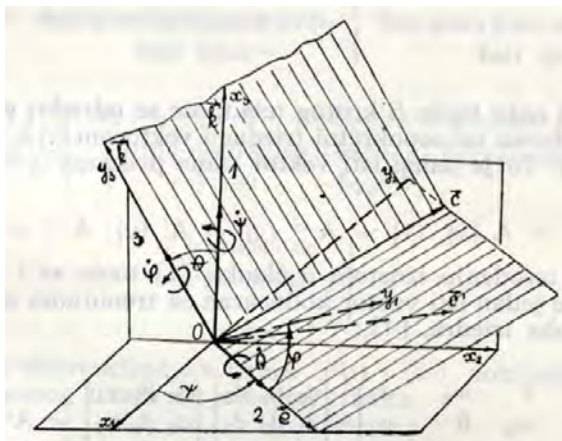


**Figure 2**. Euler's angles – generalized independent angle coordinates of three degrees of freedom of a rigid body rotating around a fixed point (Figure from paper [4])

Since, the unit vectors $\left(\vec{j}_r\right) = \left( {}_r \quad {}_r \quad {}_r \right)$, $r = 1,2,3$ are orthogonal of the respective coordinate axes, and three basic unit vectors of the orientation of the axes of coordinate system are orthogonal, there are six conditions between the matrix $\mathbf{A}$ elements:

$$\left(\vec{j}_r\right)\left\{\vec{j}_s\right\} = \left( {}_r \quad {}_r \quad {}_r \right)\begin{Bmatrix} {}_s \\ {}_s \\ {}_s \end{Bmatrix} = {}_r\,{}_s + {}_r\,{}_s + {}_r\,{}_s = \begin{cases} 1 & r = s; \qquad = \cos \\ 0 & r \neq s; \qquad = \cos \end{cases} \tag{3}$$

As far as the rigid body is concerned, it is free to have six degrees of freedom of movement, but as it is a rotation around a fixed point in which the body is fixed at fixed point, it follows that three constraints are imposed on him, that is, his number of degrees of freedom has been reduced movement to three degrees of freedoms. It follows because the binding to a fixed point eliminates three translations into three orthogonal directions, and that the body now has three degrees of freedom of movement, or three component rotations.

All this indicates that the rotation of a rigid body around a fixed point for which it has three degrees of freedom of movement expressed over three angles, so that three Euler angles can be taken instead of the preceding angles: the angle of precession  , the angle of nutrition $\vartheta$ and the angle of its own rotation $\vartheta$. Using three consecutive successive rotations, by three Euler's angles, the coordinate system $Ox_1x_2x_3$ can be moved to a moving coordinate system $Oy_1y_2y_3$:

1 * First rotation around the coordinate axis $Ox_3$ for *the precession angle*  , which is determined by the matrix $\mathbf{A}_3$, whose rows- elements are the direction cosines of the unit vectors $(\vec{e})$, $(\vec{c})$ and $(\vec{k})$ . The $Ox_1$ axis rotates and this axis go in the axis with the unit vector $(\vec{e})$, which we call the ***node axis***, while the axis $Ox_2$ has passed into the axis with the unit orientation vector $(\vec{c})$;

$$\mathbf{A}_3 = \begin{pmatrix} \cos & \sin & 0 \\ -\sin & \cos & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4}$$

2 * The rotation then proceeds around a ***node axis***, oriented by a unit vector $(\vec{e})$, for the ***angle of nutrition*** $\vartheta$, and this rotation is defined by a matrix $\mathbf{A}_2$ , whose rows - elements are the cosines of the directions of the unit vectors $(\vec{e})$, $(\vec{c})$ и $(\vec{k}')$ and measured in relation to the previous coordinate system;

$$\mathbf{A}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta \\ 0 & -\sin\vartheta & \cos\vartheta \end{pmatrix} \tag{5}$$

3 * and the third rotation around the axis $Oy_3$ for the ***angle of its own rotation*** $\vartheta$, which is determined by the matrix $\mathbf{A}_1$, whose rows  elements are the cosines of the directional directions of the unit vectors $(\vec{j}_1)$, $(\vec{j}_2)$ and $(\vec{j}_3')$, and are measured in relation to the previous coordinate system;

$$\mathbf{A}_1 = \begin{pmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6}$$

The total resulting rotation, as the sum of the previously mentioned three successive rotations, is expressed by matrix $\mathbf{A}$ of summarized rotation in the following form:

$$\mathbf{A} = \mathbf{A}_1\mathbf{A}_2\mathbf{A}_3 = \begin{pmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta \\ 0 & -\sin\vartheta & \cos\vartheta \end{pmatrix}\begin{pmatrix} \cos & \sin & 0 \\ -\sin & \cos & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$
$$= \begin{pmatrix} \cos\ \cos\vartheta - \sin\ \sin\vartheta\cos\vartheta & \sin\ \cos\vartheta + \cos\ \sin\vartheta\cos\vartheta & \sin\vartheta\sin\vartheta \\ -\cos\ \sin\vartheta - \sin\ \cos\vartheta\cos\vartheta & -\sin\ \sin\vartheta + \cos\ \cos\vartheta\cos\vartheta & \cos\vartheta\sin\vartheta \\ \sin\ \sin\vartheta & -\cos\ \sin\vartheta & \cos\vartheta \end{pmatrix} \tag{7}$$

and this presents the product between matrices $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ of three components successive rotations.

The position of a point in a rigid body can be determined by the position vector $\{x\}$ relative to the immobile coordinate system $Ox_1x_2x_3$ and a vector relative to the moving coordinate system $\{y\}$. This is the same vector only displayed in both triads' coordinate system, so it will be:

$$\{y\} = \mathbf{A}\{x\} \qquad \{x\} = \mathbf{A}^{-1}\{y\} = \mathbf{A}'\{y\} \qquad\qquad \mathbf{A}' = \mathbf{A}^{-1} \tag{8}$$

## 3. Matrix expressions of the angular velocity, angular acceleration and angular accelerating of the second order of the rigid body rotation around a fixed point

Similar to the derivation in the article (Reference by Rašković, Sokolović ]7]), a vector of total angular velocity, $(\mathit{\Omega})$ and $(\Omega)$, which is the same vector collinear with the current – instantaneous axis of rotation of rigid body around a fixed point, can be represented in matrixform in both coordinate systems, the fixed $Ox_1x_2x_3$ and the moving $Oy_1y_2y_3$ one[4]:

$$(\mathit{\Omega}) = \begin{pmatrix} 0 & -\mathit{\Omega}_3 & \mathit{\Omega}_2 \\ \mathit{\Omega}_3 & 0 & -\mathit{\Omega}_1 \\ -\mathit{\Omega}_2 & \mathit{\Omega}_1 & 0 \end{pmatrix} = \begin{pmatrix} \dot{}_1 & \dot{}_2 & \dot{}_3 \\ \dot{}_1 & \dot{}_2 & \dot{}_3 \\ \dot{}_1 & \dot{}_2 & \dot{}_3 \end{pmatrix} \begin{pmatrix} {}_1 & {}_1 & {}_1 \\ {}_2 & {}_2 & {}_2 \\ {}_3 & {}_3 & {}_3 \end{pmatrix} = \dot{\mathbf{A}}'\mathbf{A} \tag{9}$$

$$(\Omega) = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} = -\begin{pmatrix} \dot{}_1 & \dot{}_1 & \dot{}_1 \\ \dot{}_2 & \dot{}_2 & \dot{}_2 \\ \dot{}_3 & \dot{}_3 & \dot{}_3 \end{pmatrix} \begin{pmatrix} {}_1 & {}_2 & {}_3 \\ {}_1 & {}_2 & {}_3 \\ {}_1 & {}_2 & {}_3 \end{pmatrix} = -\dot{\mathbf{A}}\mathbf{A}' = \mathbf{A}\dot{\mathbf{A}}' \tag{10}$$

The preceding matrix equations (9) and (10) represent Euler's kinematic equations for the angular velocity $(\mathit{\Omega})$ as well as $(\Omega)$ in the matrix form and in the fixed $Ox_1x_2x_3$ (9) or in the movable $Oy_1y_2y_3$ (10) system of the coordinates.

By developing the previous matrix equations (9) and (10) for the instantaneous angular velocity $(\mathit{\Omega})$ as well as $(\Omega)$ of the rigid body rotation around a fixed point, we obtain Euler's kinematic equations, which represent the components of the current angular velocity rotation of a rigid body around a fixed point in the fixed $Ox_1x_2x_3$ or in the movable $Oy_1y_2y_3$ coordinate system [2]:

$$\begin{aligned} \mathit{\Omega}_1 &= \dot{\vartheta}\cos\ \ + \vartheta\sin\vartheta\sin & \qquad \Omega_1 &= \dot{\vartheta}\cos\vartheta + \dot{}\ \sin\vartheta\sin\vartheta \\ \mathit{\Omega}_2 &= \dot{\vartheta}\sin\ \ - \vartheta\sin\vartheta\cos & \qquad \Omega_2 &= -\dot{\vartheta}\sin\vartheta + \dot{}\ \sin\vartheta\cos\vartheta \\ \mathit{\Omega}_3 &= \dot{}\ + \vartheta\cos\vartheta & \qquad \Omega_3 &= \vartheta + \dot{}\ \cos\vartheta \end{aligned} \tag{11}$$

However, these Euler's kinematic equations can be easily obtained by a vector-matrix relation, expressing them by means of the derivation of the Euler angles by time and by the unit vector of the node axis orientation $(\vec{e})$, the axis of the precession $(\vec{k})$ and the figurative axis $(\vec{k}')$:

$$\{\vec{\mathit{\Omega}}\} = \dot{\vartheta}\{\vec{e}\} + \dot{}\ \{\vec{k}\} + \vartheta\{\vec{k}'\} = \{\vec{\Omega}\} \tag{12}$$

$$\mathit{\Omega}^2 = \dot{\vartheta}^2 + \dot{}\ ^2 + \vartheta^2 + 2\vartheta\ \dot{}\ \cos\vartheta \tag{13}$$

or in a expanded form:

$$\{\Omega\}=\begin{Bmatrix}\Omega_1\\ \Omega_{22}\\ \Omega_3\end{Bmatrix}=\begin{pmatrix}\cos & 0 & \sin\vartheta\sin\\ \sin & 0 & -\sin\vartheta\cos\\ 0 & 1 & \cos\vartheta\end{pmatrix}\begin{Bmatrix}\dot{\vartheta}\\ \cdot\\ \vartheta\end{Bmatrix}=\mathbf{S}\begin{Bmatrix}\dot{\vartheta}\\ \cdot\\ \vartheta\end{Bmatrix} \tag{14}$$

$$\{\Omega\}=\begin{Bmatrix}\Omega_1\\ \Omega_{22}\\ \Omega_3\end{Bmatrix}=\mathbf{S}\begin{Bmatrix}\dot{\vartheta}\\ \cdot\\ \vartheta\end{Bmatrix}=\begin{pmatrix}\cos & 0 & \sin\vartheta\sin\\ \sin & 0 & -\sin\vartheta\cos\\ 0 & 1 & \cos\vartheta\end{pmatrix}\begin{Bmatrix}\dot{\vartheta}\\ \cdot\\ \vartheta\end{Bmatrix}$$

$$\{\vec{\Omega}\}=\begin{Bmatrix}\Omega_1\\ \Omega_{22}\\ \Omega_3\end{Bmatrix}=\begin{pmatrix}\cos\vartheta & \sin\vartheta\sin\vartheta & o\\ -\sin\vartheta & \sin\vartheta\cos\vartheta & 0\\ 0 & \cos\vartheta & 1\end{pmatrix}\begin{Bmatrix}\dot{\vartheta}\\ \cdot\\ \vartheta\end{Bmatrix}=\mathbf{P}\begin{Bmatrix}\dot{\vartheta}\\ \cdot\\ \vartheta\end{Bmatrix} \tag{15}$$

$$\{\vec{\Omega}\}=\begin{Bmatrix}\Omega_1\\ \Omega_{22}\\ \Omega_3\end{Bmatrix}=\mathbf{P}\begin{Bmatrix}\dot{\vartheta}\\ \cdot\\ \vartheta\end{Bmatrix}=\begin{pmatrix}\cos\vartheta & \sin\vartheta\sin\vartheta & o\\ -\sin\vartheta & \sin\vartheta\cos\vartheta & 0\\ 0 & \cos\vartheta & 1\end{pmatrix}\begin{Bmatrix}\dot{\vartheta}\\ \cdot\\ \vartheta\end{Bmatrix}$$

where:

$$\mathbf{S}=\begin{pmatrix}\cos & 0 & \sin\vartheta\sin\\ \sin & 0 & -\sin\vartheta\cos\\ 0 & 1 & \cos\vartheta\end{pmatrix} \tag{16}$$

$$\mathbf{P}=\begin{pmatrix}\cos\vartheta & \sin\vartheta\sin\vartheta & o\\ -\sin\vartheta & \sin\vartheta\cos\vartheta & 0\\ 0 & \cos\vartheta & 1\end{pmatrix} \tag{17}$$

Matrices $\mathbf{S}$ and $\mathbf{P}$, defined by (16) and (17) are the matrices of single column of unit vectors of *the node axis* $(\vec{e})$, the axis of *the precession* $(\vec{k})$ and *the figurative axis* $(\vec{k}')$, measured in relation to the immobile coordinate system $Ox_1x_2x_3$ and in relation to the moving triad-moving coordinate system $Oy_1y_2y_3$.

Now, by differentiating the previous expressions (14) and (15) for the angular velocity, $(\Omega)$ as well as $(\Omega)$, of rotation around the fixed point over time, projections of angular acceleration, $(\Omega)$ as well as $(\dot{\Omega})$, are obtained on the axes of both triads of the fixed $Ox_1x_2x_3$ or in the movable $Oy_1y_2y_3$ coordinate system in the following form:

1* Projections of angular acceleration $(\Omega)$, in immobile triad-immobile coordinate system $Ox_1x_2x_3$, of a rigid body that rotates around a fixed point $O$:

$$\{\dot{\Omega}\}=\begin{Bmatrix}\Omega_1\\ \Omega_{22}\\ \Omega_3\end{Bmatrix}=\mathbf{S}\begin{Bmatrix}\ddot{\vartheta}\\ \cdot\cdot\\ \ddot{\vartheta}\end{Bmatrix}+\dot{\mathbf{S}}\begin{Bmatrix}\dot{\vartheta}\\ \cdot\\ \vartheta\end{Bmatrix}$$

$$\begin{Bmatrix} \Omega_1 \\ \Omega_{22} \\ \Omega_3 \end{Bmatrix} = \begin{pmatrix} \cos & 0 & \sin\vartheta\sin \\ \sin & 0 & -\sin\vartheta\cos \\ 0 & 1 & \cos\vartheta \end{pmatrix} \begin{Bmatrix} \ddot{\vartheta} \\ \ddot{} \\ \dot{\vartheta} \end{Bmatrix} + \frac{d}{dt} \begin{pmatrix} \cos & 0 & \sin\vartheta\sin \\ \sin & 0 & -\sin\vartheta\cos \\ 0 & 1 & \cos\vartheta \end{pmatrix} \begin{Bmatrix} \ddot{\vartheta} \\ \ddot{} \\ \dot{\vartheta} \end{Bmatrix} \tag{18}$$

$$\{\dot{\Omega}\} = \begin{Bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{Bmatrix} = \mathbf{S} \begin{Bmatrix} \ddot{\vartheta} \\ \ddot{} \\ \dot{\vartheta} \end{Bmatrix} + \dot{\mathbf{S}} \begin{Bmatrix} \dot{\vartheta} \\ \dot{} \\ \vartheta \end{Bmatrix}$$

$$\begin{Bmatrix} \Omega_1 \\ \Omega_{22} \\ \Omega_3 \end{Bmatrix} = \begin{Bmatrix} \ddot{\vartheta}\cos \;+ \dot{\vartheta}\sin\vartheta\sin \;- \dot{\vartheta}\,\dot{}\,\sin \;+ \dot{\vartheta}\dot{\vartheta}\cos\vartheta\sin \;+ \,\dot{}\,\dot{\vartheta}\sin\vartheta\cos \\ \ddot{\vartheta}\sin \;- \dot{\vartheta}\sin\vartheta\cos \;+ \dot{\vartheta}\,\dot{}\,\cos \;- \dot{\vartheta}\dot{\vartheta}\cos\vartheta\cos \;+ \,\dot{}\,\dot{\vartheta}\sin\vartheta\sin s \\ \ddot{} + \dot{\vartheta}\cos\vartheta - \dot{\vartheta}\dot{\vartheta}\sin\vartheta \end{Bmatrix} \tag{19}$$

2* Projections of angular acceleration $(\dot{\Omega})$, in mobile triad-mobile coordinate system $Oy_1y_2y_3$, of a rigid body that rotates around a fixed point $O$:

$$\{\dot{\Omega}\} = \begin{Bmatrix} \dot{\Omega}_1 \\ \dot{\Omega}_{22} \\ \dot{\Omega}_3 \end{Bmatrix} = \mathbf{P} \begin{Bmatrix} \ddot{\vartheta} \\ \ddot{} \\ \dot{\vartheta} \end{Bmatrix} + \dot{\mathbf{P}} \begin{Bmatrix} \dot{\vartheta} \\ \dot{} \\ \vartheta \end{Bmatrix}$$

$$\{\dot{\Omega}\} = \begin{Bmatrix} \dot{\Omega}_1 \\ \dot{\Omega}_{22} \\ \dot{\Omega}_3 \end{Bmatrix} = \begin{pmatrix} \cos\vartheta & \sin\vartheta\sin\vartheta & o \\ -\sin\vartheta & \sin\vartheta\cos\vartheta & 0 \\ 0 & \cos\vartheta & 1 \end{pmatrix} \begin{Bmatrix} \ddot{\vartheta} \\ \ddot{} \\ \dot{\vartheta} \end{Bmatrix} + \frac{d}{dt} \begin{pmatrix} \cos\vartheta & \sin\vartheta\sin\vartheta & o \\ -\sin\vartheta & \sin\vartheta\cos\vartheta & 0 \\ 0 & \cos\vartheta & 1 \end{pmatrix} \begin{Bmatrix} \dot{\vartheta} \\ \dot{} \\ \vartheta \end{Bmatrix} \tag{20}$$

$$\begin{Bmatrix} \dot{\Omega}_1 \\ \dot{\Omega}_{22} \\ \dot{\Omega}_3 \end{Bmatrix} = \begin{Bmatrix} \ddot{\vartheta}\cos\vartheta + \ddot{}\sin\vartheta\sin\vartheta - \dot{\vartheta}\dot{\vartheta}\sin\vartheta - \dot{\vartheta}\,\dot{}\,\cos\vartheta\sin\vartheta + \,\dot{}\,\dot{\vartheta}\sin\vartheta\cos\vartheta \\ -\ddot{\vartheta}\sin\vartheta + \ddot{}\sin\vartheta\cos\vartheta + \dot{\vartheta}\dot{\vartheta}\cos\vartheta - \dot{\vartheta}\,\dot{}\,\cos\vartheta\cos\vartheta - \,\dot{}\,\dot{\vartheta}\sin\vartheta\sin\vartheta \\ \dot{\vartheta} + \ddot{}\cos\vartheta - \dot{\vartheta}\,\dot{}\,\sin\vartheta \end{Bmatrix} \tag{21}$$

By again differentiating the preceding expressions (19) and (21) for the angular acceleration, $(\Omega)$ and $(\dot{\Omega})$, determined in the motionless and moving triad, we obtain projections of a vector of the angle jerk-the angular acceleration of the second order $\{\ddot{\Omega}\}$ and $\{\ddot{\Omega}\}$ of the kinematics of a rigid body which rotates around a fixed point $O$, both in the immobile $Ox_1x_2x_3$ and the moving $Oy_1y_2y_3$ triad - coordinate system.

1* Projections of angular acceleration of the second order $\{\ddot{\Omega}\}$, in immobile triad-immobile coordinate system $Ox_1x_2x_3$, of a rigid body that rotates around a fixed point $O$ is in the form:

$$\{\dot{\Omega}\} = \begin{Bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{Bmatrix} = \mathbf{S} \begin{Bmatrix} \ddot{\vartheta} \\ \ddot{} \\ \dot{\vartheta} \end{Bmatrix} + \dot{\mathbf{S}} \begin{Bmatrix} \dot{\vartheta} \\ \dot{} \\ \vartheta \end{Bmatrix}$$

$$\{\ddot{\Omega}\} = \begin{Bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{Bmatrix} = \mathbf{S} \begin{Bmatrix} \dddot{\vartheta} \\ \dddot{} \\ \ddot{\vartheta} \end{Bmatrix} + 2\dot{\mathbf{S}} \begin{Bmatrix} \ddot{\vartheta} \\ \ddot{} \\ \dot{\vartheta} \end{Bmatrix} + \ddot{\mathbf{S}} \begin{Bmatrix} \dot{\vartheta} \\ \dot{} \\ \vartheta \end{Bmatrix} \tag{22}$$

By differentiating the preceding expression (16) for $\mathbf{S}$ and by simplifying expression (22), projections of angular acceleration of the second order $\{\ddot{\vec{\Omega}}\}$ in scalar form can be obtain, in immobile triad-immobile coordinate system $Ox_1x_2x_3$, of a rigid body that rotates around a fixed point $O$, in the following forms:

$$\begin{aligned}\Omega_1 &= \left(\dddot{\vartheta} - \dot{\vartheta}\,\dot{}^{\,2}\right)\cos \;\; - \left(2\ddot{\vartheta}\,\dot{} + \dot{\vartheta}\,\dddot{}\right)\sin \;\; + \left(\ddot{\vartheta} - \dot{\vartheta}^2\ddot{\vartheta} - \dot{}^{\,2}\ddot{\vartheta}\right)\sin\vartheta\sin \;\; + \\ &+ \left(2\ddot{\vartheta}\dot{\vartheta} + \dot{\vartheta}\ddot{\vartheta}\right)\cos\vartheta\sin \;\; + \left(2\,\dot{}\,\ddot{\vartheta} + \ddot{}\,\vartheta\right)\sin\vartheta\cos \;\; + 2\,\dot{}\,\vartheta\dot{\vartheta}\cos\vartheta\cos\end{aligned} \tag{23}$$

$$\begin{aligned}\Omega_2 &= \left(\dddot{\vartheta} - \dot{\vartheta}\,\dot{}^{\,2}\right) sim \;\; - \left(2\ddot{\vartheta}\,\dot{} + \dot{\vartheta}\,\dddot{}\right)\cos \;\; - \left(\ddot{\vartheta} - \dot{\vartheta}^2\ddot{\vartheta} - \dot{}^{\,2}\ddot{\vartheta}\right)\sin\vartheta\cos \;\; - \\ &- \left(2\ddot{\vartheta}\dot{\vartheta} + \dot{\vartheta}\ddot{\vartheta}\right)\cos\vartheta\cos \;\; + \left(2\,\dot{}\,\ddot{\vartheta} + \ddot{}\,\vartheta\right)\sin\vartheta\sin \;\; + 2\,\dot{}\,\vartheta\dot{\vartheta}\cos\vartheta\sin\end{aligned} \tag{24}$$

$$\Omega_3 = \dddot{} + \left(\ddot{\vartheta} - \dot{\vartheta}^2\ddot{\vartheta}\right)\cos\vartheta - \left(2\ddot{\vartheta}\dot{\vartheta} + \vartheta\ddot{\vartheta}\right)\sin\vartheta \tag{25}$$

2* Projections of angular acceleration of the second order $\{\ddot{\vec{\Omega}}\}$, in mobile triad-mobile coordinate system $Oy_1y_2y_3$, of a rigid body that rotates around a fixed point $O$ is in the form:

$$\{\dot{\vec{\Omega}}\} = \begin{Bmatrix}\dot{\Omega}_1 \\ \dot{\Omega}_{22} \\ \dot{\Omega}_3\end{Bmatrix} = \mathbf{P}\begin{Bmatrix}\ddot{\vartheta} \\ \ddot{} \\ \ddot{\vartheta}\end{Bmatrix} + \dot{\mathbf{P}}\begin{Bmatrix}\dot{\vartheta} \\ \dot{} \\ \dot{\vartheta}\end{Bmatrix}$$

$$\{\ddot{\vec{\Omega}}\} = \begin{Bmatrix}\ddot{\Omega}_1 \\ \ddot{\Omega}_{22} \\ \ddot{\Omega}_3\end{Bmatrix} = \mathbf{P}\begin{Bmatrix}\dddot{\vartheta} \\ \dddot{} \\ \dddot{\vartheta}\end{Bmatrix} + 2\dot{\mathbf{P}}\begin{Bmatrix}\ddot{\vartheta} \\ \ddot{} \\ \ddot{\vartheta}\end{Bmatrix} + \ddot{\mathbf{P}}\begin{Bmatrix}\dot{\vartheta} \\ \dot{} \\ \dot{\vartheta}\end{Bmatrix} \tag{26}$$

By differentiating the preceding expression (17) for $\mathbf{P}$ and by simplification expression (26), projections of angular acceleration of the second order $\{\ddot{\vec{\Omega}}\}$ in scalar form can be obtain, in mobile triad-mobile coordinate system $Oy_1y_2y_3$, of a rigid body that rotates around a fixed point $O$, in the following forms:

$$\begin{aligned}\ddot{\Omega}_1 &= \left(\dddot{\vartheta} - \dot{\vartheta}\dot{\vartheta}^2\right)\cos\vartheta - \left(2\ddot{\vartheta}\dot{\vartheta} + \dot{\vartheta}\ddot{\vartheta}\right)\sin\vartheta + \left(\dddot{} - \dot{\vartheta}^2\,\dot{} - \dot{}\,\dot{\vartheta}^2\right)\sin\vartheta\sin\vartheta + \\ &+ \left(2\,\ddot{}\,\dot{\vartheta} + \dot{}\,\ddot{\vartheta}\right)\cos\vartheta\sin\vartheta + \left(2\,\ddot{}\,\dot{\vartheta} + \dot{}\,\ddot{\vartheta}\right)\sin\vartheta\cos\vartheta + 2\,\dot{}\,\vartheta\dot{\vartheta}\cos\vartheta\cos\vartheta\end{aligned} \tag{27}$$

$$\begin{aligned}\ddot{\Omega}_2 &= -\left(\dddot{\vartheta} - \dot{\vartheta}\dot{\vartheta}^2\right)\sin\vartheta - \left(2\ddot{\vartheta}\dot{\vartheta} + \dot{\vartheta}\ddot{\vartheta}\right)\cos\vartheta + \left(\dddot{} - \dot{\vartheta}^2\,\dot{} - \dot{}\,\dot{\vartheta}^2\right)\sin\vartheta\cos\vartheta + \\ &+ \left(2\,\ddot{}\,\dot{\vartheta} + \dot{}\,\ddot{\vartheta}\right)\cos\vartheta\cos\vartheta - \left(2\,\ddot{}\,\dot{\vartheta} + \dot{}\,\ddot{\vartheta}\right)\sin\vartheta\sin\vartheta - 2\,\dot{}\,\vartheta\dot{\vartheta}\cos\vartheta\sin\vartheta\end{aligned} \tag{28}$$

$$\ddot{\Omega}_3 = \ddot{\vartheta} + \left(\dddot{} - \dot{\vartheta}^2\,\dot{}\right)\cos\vartheta - \left(2\,\ddot{}\,\dot{\vartheta} + \dot{}\,\ddot{\vartheta}\right)\sin\vartheta \tag{29}$$

4. **Matrix and scalar expressions of the acceleration of the second order of a point of the rigid body rotation around a fixed point**

By differentiating the preceding equation $\{x\} = \mathbf{A}'\{y\}$ and taking into account that $\{\dot{y}\} = 0$, we can obtain the vector $\{\ddot{x}\}$ of the acceleration of a point point $B$, $\overrightarrow{OB} = \vec{r}(x_1, x_2, x_3$

of the rigid body rotation around fixed point $O$ in immobile coordinate system $Ox_1x_2x_3$ in the following form:

$$\{\dot{x}\} = \dot{\mathbf{A}}'\{y\} = \dot{\mathbf{A}}'\mathbf{A}'\{x\} = [\Omega]\{x\} = \{\Omega\} \tag{30}$$

$$\mathbf{A}\{\dot{x}\} = \mathbf{A}\dot{\mathbf{A}}'\{y\} = [\Omega]\{y\} = \{v\} \tag{31}$$

$$\{\dot{x}\} = [\Omega]\{x\} = \{\Omega\}$$

$$\{\ddot{x}\} = [\dot{\Omega}]\{x\} + [\Omega]\{\dot{x}\} = \left([\dot{\Omega}] + [\Omega]^2\right)\{x\} \tag{32}$$

From last expressions (30), (31) and (32), it can be seen that there is a qualitative and mathematical analogy: matrix $[\Omega]$ can be changed by matrix $[\Omega]$ as well as matrix $\{x\}$ by matrix $\{y\}$.

Next by differentiating the preceding equation (32) and taking into account that (30), we can obtain the vector $\{\dddot{x}\}$ of the acceleration of the second order of a point $B$, $\overrightarrow{OB} = \vec{r}(x_1, x_2, x_3$ of the rigid body rotation around the fixed point $O$ in the immobile coordinate system $Ox_1x_2x_3$ in the following form:

$$\{\dddot{x}\} = \left([\ddot{\Omega}] + [\dot{\Omega}][\Omega] + [\Omega][\dot{\Omega}]\right)\{x\} + \left([\dot{\Omega}] + [\Omega]^2\right)\{\dot{x}\}$$

$$\{\dddot{x}\} = \left([\ddot{\Omega}] + [\dot{\Omega}][\Omega] + [\Omega][\dot{\Omega}]\right)\{x\} + \left([\dot{\Omega}] + [\Omega]^2\right)[\Omega]\{x\}$$

$$\{\dddot{x}\} = \left([\ddot{\Omega}] + 2[\dot{\Omega}][\Omega] + [\Omega][\dot{\Omega}] + [\Omega]^3\right)\{x\}$$

$$\{\dddot{x}\} = \left([\ddot{\Omega}] + 2[\dot{\Omega}][\Omega] + [\Omega][\dot{\Omega}] - \Omega^2[\Omega]\right)\{x\} \tag{33}$$

After presenting the necessary matrix expressions and obtaining necessary matrix products in the forms:

$$[\dot{\Omega}] = \begin{pmatrix} 0 & -\dot{\Omega}_3 & \dot{\Omega}_2 \\ \dot{\Omega}_3 & 0 & -\dot{\Omega}_1 \\ -\dot{\Omega}_2 & \dot{\Omega}_1 & 0 \end{pmatrix} \tag{34}$$

$$[\dot{\Omega}][\Omega] = \begin{pmatrix} 0 & -\dot{\Omega}_3 & \dot{\Omega}_2 \\ \dot{\Omega}_3 & 0 & -\dot{\Omega}_1 \\ -\dot{\Omega}_2 & \dot{\Omega}_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}$$

$$[\dot{\Omega}][\Omega] = \begin{pmatrix} -\Omega_2\dot{\Omega}_2 - \Omega_3\dot{\Omega}_3 & \dot{\Omega}_1\Omega_2 & \dot{\Omega}_1\Omega_3 \\ \dot{\Omega}_1\Omega_2 & -\Omega_3\dot{\Omega}_3 - \Omega_1\dot{\Omega}_1 & \dot{\Omega}_2\Omega_3 \\ \dot{\Omega}_1\Omega_3 & \dot{\Omega}_2\Omega_3 & -\Omega_1\dot{\Omega}_1 - \Omega_2\dot{\Omega}_2 \end{pmatrix} \tag{35}$$

$$[\Omega][\dot{\Omega}] = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\dot{\Omega}_3 & \dot{\Omega}_2 \\ \dot{\Omega}_3 & 0 & -\dot{\Omega}_1 \\ -\dot{\Omega}_2 & \dot{\Omega}_1 & 0 \end{pmatrix}$$

$$[\Omega][\dot{\Omega}] = \begin{pmatrix} -\Omega_2\dot{\Omega}_2 - \Omega_3\dot{\Omega}_3 & \dot{\Omega}_1\Omega_2 & \dot{\Omega}_1\Omega_3 \\ \Omega_1\dot{\Omega}_2 & -\Omega_3\dot{\Omega}_3 - \Omega_1\dot{\Omega}_1 & \dot{\Omega}_2\Omega_3 \\ \Omega_1\dot{\Omega}_3 & \Omega_2\dot{\Omega}_3 & -\Omega_1\dot{\Omega}_1 - \Omega_2\dot{\Omega}_2 \end{pmatrix} \tag{36}$$

$$\Omega^2[\Omega] = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}$$

$$\Omega^2[\Omega] = \begin{pmatrix} \Omega_1^2 - \Omega^2 & \Omega_1\Omega_2 & \Omega_1\Omega_3 \\ \Omega_1\Omega_2 & \Omega_2^2 - \Omega^2 & \Omega_2\Omega_3 \\ \Omega_1\Omega_3 & \Omega_2\Omega_3 & \Omega_3^2 - \Omega^2 \end{pmatrix}\begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} \tag{37}$$

and introducing expressions (36) and (36) into the equation (35) we can obtain projections $\dddot{x}_1$, $\dddot{x}_2$ and $\dddot{x}_3$ of the vector $\{\dddot{x}\}$ or $\vec{w}$ of the acceleration of the second order of a point $B$, $\overrightarrow{OB} = \vec{r}(x_1, x_2, x_3$ of the rigid body rotation around fixed point $O$ in immobile coordinate system $Ox_1x_2x_3$ in the following scalar form:

$$\begin{aligned} \dddot{x}_1 = {} & (\ddot{\Omega}_2 x_3 - \ddot{\Omega}_3 x_2) + 2\,\Omega_1(\dot{\Omega}_2 x_2 + \dot{\Omega}_3 x_3) - 3(\Omega_2\dot{\Omega}_2 + \Omega_3\dot{\Omega}_3)x_1 + \\ & + \dot{\Omega}_1(\Omega_2 x_2 + \Omega_3 x_3) - \Omega^2(\Omega_2 x_3 - \Omega_3 x_2) \end{aligned} \tag{38}$$

$$\begin{aligned} \dddot{x}_2 = {} & (\ddot{\Omega}_3 x_1 - \ddot{\Omega}_1 x_3) + 2\,\Omega_2(\dot{\Omega}_1 x_1 + \dot{\Omega}_3 x_3) - 3(\Omega_1\dot{\Omega}_1 + \Omega_3\dot{\Omega}_3)x_2 + \\ & + \dot{\Omega}_2(\Omega_2 x_2 + \Omega_1 x_1) - \Omega^2(\Omega_1 x_3 + \Omega_3 x_1) \end{aligned} \tag{39}$$

$$\begin{aligned} \dddot{x}_3 = {} & (\ddot{\Omega}_1 x_2 - \ddot{\Omega}_2 x_3) + 2\,\Omega_3(\dot{\Omega}_2 x_2 + \dot{\Omega}_1 x_1) - 3(\Omega_2\dot{\Omega}_2 + \Omega_1\dot{\Omega}_1)x_3 + \\ & + \dot{\Omega}_3(\Omega_2 x_2 + \Omega_1 x_1) - \Omega^2(\Omega_1 x_2 + \Omega_2 x_1) \end{aligned} \tag{40}$$

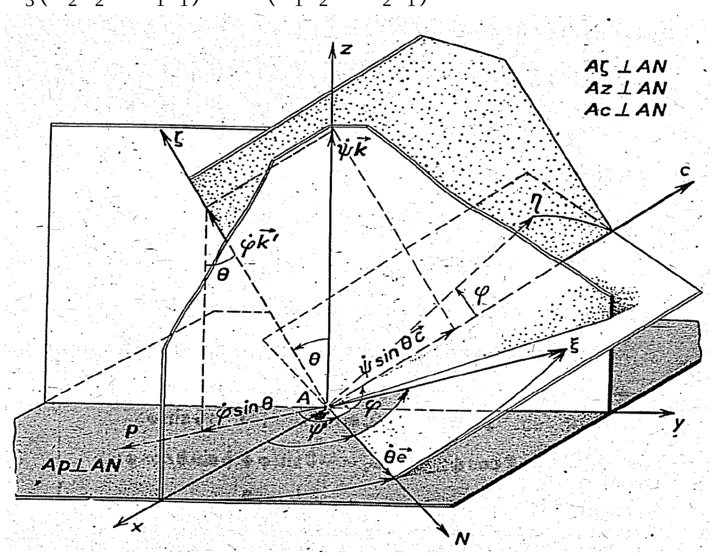



**Figure 2.** Components of the vector $\{\Omega\} = \dot{\vartheta}\{\vec{e}\} + \dot{}\{\vec{k}\} + \vartheta\{\vec{k}'\} = \{\vec{\Omega}\}$ of angular velocity of a rigid body rotating around a fixed point $A$ (Figure from text book [1])

Multiplying equation (32) by matrix **A** in the form (7) we can obtain the vector $\vec{w}$ or matrix $\{w\}$ of the acceleration of the second order of a point $B$, $\overrightarrow{OB} = \vec{r}(y_1, y_2, y_3$ of the rigid body rotation around fixed point $O$ in the mobile coordinate system $Oy_1y_2y_3$ in the following matrix form:

$$\mathbf{A}\{\ddot{x}\} = \mathbf{A}\left([\ddot{\Omega}] + 2[\dot{\Omega}][\Omega] + [\Omega][\dot{\Omega}] - \Omega^2[\Omega]\right)\{x\}$$
$$\{w\} = \mathbf{A}\{\ddot{x}\} = \left([\ddot{\Omega}] + 2[\Omega][\dot{\Omega}] + [\dot{\Omega}][\Omega] - \Omega^2[\Omega]\right)\{y\} \qquad (41)$$

From previous matrix equation (41) we can obtain projections $w_1$, $w_2$ and $w$ of the vector $\{w\}$ or $\vec{w}$ of the acceleration of the second order of a point $B$, $\overrightarrow{OB} = \vec{r}(y_1, y_2, y_3$ of the rigid body rotation around the fixed point $O$ in the mobile coordinate system $Oy_1y_2y_3$ in the following scalar form:

$$\begin{aligned} w_1 = &\left(\ddot{\Omega}_2 y_3 - \ddot{\Omega}_3 y_2\right) + 2\dot{\Omega}_1\left(\Omega_2 y_2 + \Omega_3 y_3\right) - 3\left(\Omega_2\dot{\Omega}_2 + \Omega_3\dot{\Omega}_3\right)y_1 + \\ &+ \Omega_1\left(\dot{\Omega}_2 y_2 + \dot{\Omega}_3 y_3\right) - \Omega^2\left(\Omega_2 y_3 - \Omega_3 y_2\right) \end{aligned} \qquad (42)$$

$$\begin{aligned} w_2 = &\left(\ddot{\Omega}_3 y_1 - \ddot{\Omega}_1 y_3\right) + 2\dot{\Omega}_2\left(\Omega_1 y_1 + \Omega_3 y_3\right) - 3\left(\Omega_1\dot{\Omega}_1 + \Omega_3\dot{\Omega}_3\right)y_2 + \\ &+ \Omega_2\left(\dot{\Omega}_1 y_1 + \dot{\Omega}_3 y_3\right) - \Omega^2\left(\Omega_3 y_1 - \Omega_1 y_3\right) \end{aligned} \qquad (43)$$

$$\begin{aligned} w_3 = &\left(\ddot{\Omega}_1 y_2 - \ddot{\Omega}_2 y_1\right) + 2\dot{\Omega}_3\left(\Omega_1 y_1 + \Omega_2 y_2\right) - 3\left(\Omega_1\dot{\Omega}_1 + \Omega_2\dot{\Omega}_2\right)y_3 + \\ &+ \Omega_3\left(\dot{\Omega}_1 y_1 + \dot{\Omega}_2 y_2\right) - \Omega^2\left(\Omega_1 y_2 - \Omega_2 y_1\right) \end{aligned} \qquad (44)$$

## 5. The natural coordinates of the vector of acceleration of the second order of a rigid body rotating around a fixed point

According to Rivals's theorem, an acceleration vector of a point determined by the position vector $\vec{\rho}$, of a rigid body rotating around a fixed point with instantaneous angular velocity $\vec{\Omega}$, can be decomposed into the natural components: the tangential $\vec{a}_T$, normal - axipetal $\vec{a}_N$, and supplemental additional component $\vec{a}_d$ of a point $B$ of the rigid body that rotates around a fixed point $O$, from which we measure the help point vector for which the acceleration is determined (see Figure 3.a*, b* and c*):

$$\vec{a} = \left[\frac{d\vec{\Omega}}{dt}, \vec{\rho}\right] + \left[\vec{\Omega}, \left[\vec{\Omega}, \vec{\rho}\right]\right] = \dot{\Omega}\left[\vec{u}, \vec{\rho}\right] + \left[\vec{\Omega}, \left[\vec{\Omega}, \vec{\rho}\right]\right] - \left[\vec{\rho}, \left[\bar{\Omega}, \vec{\Omega}\right]\right] = \vec{a}_N + \vec{a}_T + \vec{a}_d \qquad (45)$$

because the angular velocity $\vec{\Omega}$ of the rigid body rotating around fixed point $O$ is changeable in magnitude $|\vec{\Omega}| = \Omega$ and in orientation by unit vector $\vec{u}$, and then its derivative $\frac{d\vec{\Omega}}{dt}$, is:

$$\frac{d\vec{\Omega}}{dt} = \frac{d(\Omega\vec{u})}{dt} = \dot{\Omega}\vec{u} + \Omega\left[\bar{\Omega}, \vec{u}\right] = \dot{\Omega}\vec{u} + \left[\bar{\Omega}, \vec{\Omega}\right] \qquad (46)$$

In previous expression $\bar{\Omega}$ is the angular velocity of the rotation of the instantaneous axis, oriented by the unit vector $\vec{u}$, of rotation of the considered rigid body when rotating around a fixed point $O$.

Since the angular velocity vector $\vec{\Omega}$ changes with both intensity $|\vec{\Omega}| = \Omega$ and direction, oriented by unit vector $\vec{u}$, this angular acceleration $\frac{d\Omega}{dt}$ of the rigid body, which rotates around a fixed point $O$, will be in the form (46).

By differentiating the preceding vector expression (46 ) for angular acceleration $\frac{d\Omega}{dt}$, we obtain a vector of the angular acceleration of the second order $\frac{d^2\Omega}{dt^2}$ of a rigid body that rotates about a fixed point $O$, in the form:

$$\frac{d^2\Omega}{dt^2} = \left(\Omega - \Omega\Omega^2\right)\vec{u} + 2\Omega\left[\vec{\Omega}, \vec{u}\right] + \left[\dot{\vec{\Omega}}, \Omega\right] + \left(\Omega, \vec{\Omega}\right)\vec{\Omega} \tag{47}$$

where $\vec{\Omega}$ and $\dot{\vec{\Omega}}$, are the angular velocity and angular acceleration of the rotating of the current instantaneous axis of rotation of a rigid body, which rotates around a fixed point $O$, and around a instantaneous axis in space, passing through a fixed point $O$ of the rigid body.

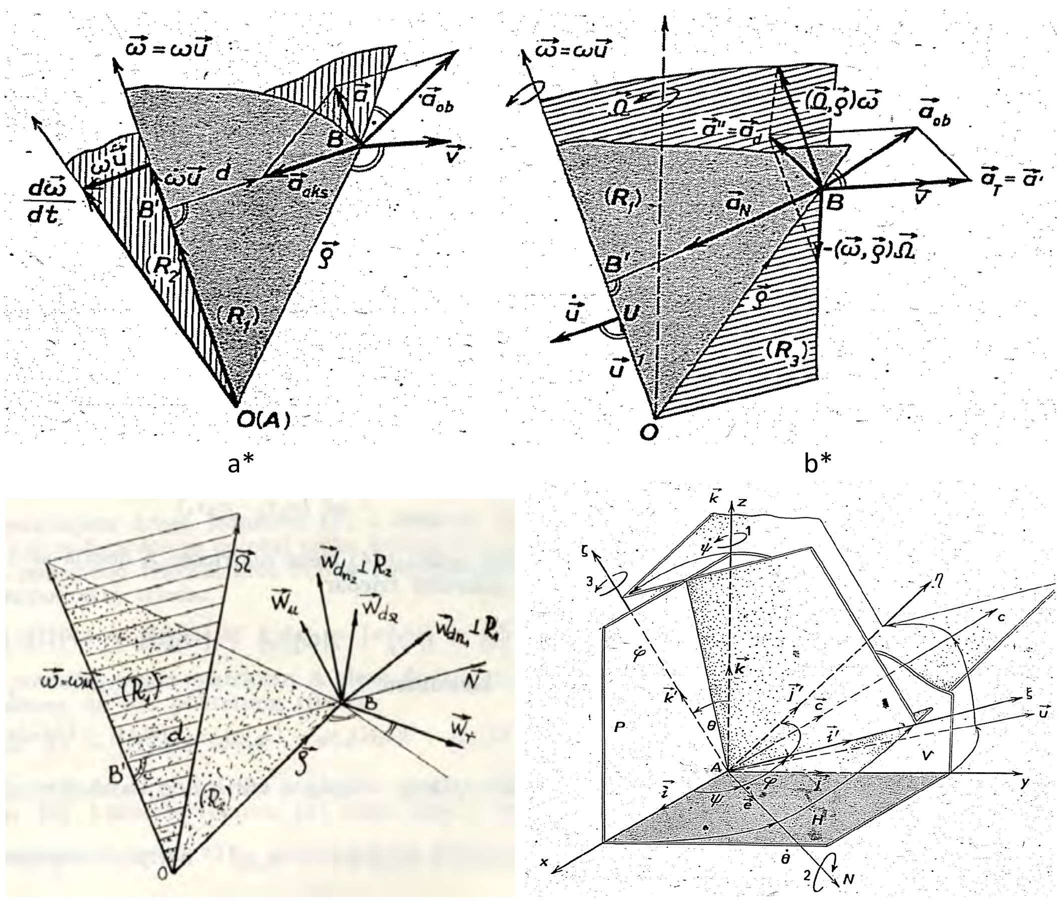



**Figure 3. a*, b*** and d* Components of the angular velocity, angular acceleration and angular acceleration of the second order and a* velocity, b* acceleration and c* acceleration of the second order of a body point of a rigid body rotating around a fixed point (Figures a*,b* and d* from Reference [1] and c* from Reference [4])

Since the angular acceleration vector $\frac{d\Omega}{dt}$ changes with both intensity and direction, and with changes of all terms in expression (46) oriented by unit vector $\vec{u}$, this angular acceleration of second order $\frac{d^2\Omega}{dt^2}$ of the rigid body, which rotates around a fixed point $O$, will be in the form (47).

By differentiating the preceding vector expression (45) for acceleration $\vec{a}$ of a point of the rigid body, we can obtain a vector of the acceleration of the second order $\vec{w} = \frac{d^2\vec{v}}{dt^2} = \frac{d\vec{a}}{dt}$ of the point $B$ of a rigid body that rotating about a fixed point $O$, in the form:

$$\vec{w} = \left[\frac{d^2\Omega}{dt^2}, \vec{\,}\right] + 2\left(\vec{\,}, \frac{d\Omega}{dt}\right)\Omega - 3\left(\Omega, \frac{d\Omega}{dt}\right)\vec{\,} + (\Omega, \vec{\,})\frac{d\Omega}{dt} - \Omega^2[\Omega, \vec{\,}] \tag{48}$$

where $\vec{\,} = \overrightarrow{OB}$ is the point position vector of the point $B$ of the rigid body when rotating around a fixed point $O$, with the instantaneous angular velocity $\Omega$.

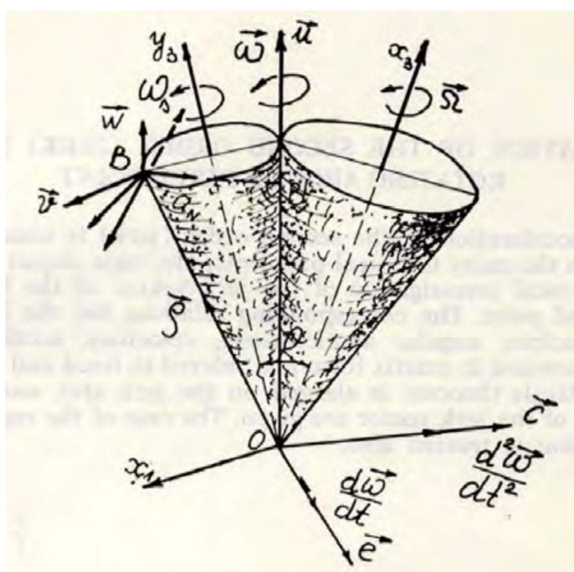


**Figure 4.** Components of the angular velocity, angular acceleration and angular acceleration of the second order and velocity, acceleration and acceleration of the second order of a body point of a rigid body rotating around a fixed point in regime of regular progressive precession motion and axoids (polhodia and herpolhodia) (Figure from Reference [4])

Let us mark with the normal distance $\overline{BB'} = d$ of the body point $B$ from the current instantaneous axis of rotation of the rigid body (Figure 3.a* and c*) rotates with instantaneous angular velocity $\Omega$, then the unit vectors of the tangent and the normal directions in relation to the current instantaneous axis of rotation are defined:

$$[\vec{u},\vec{\ }]=d\vec{T}\,,\qquad \vec{T}=\frac{1}{d}[\vec{u},\vec{\ }],\qquad \vec{\ }=\left(\vec{\ },\vec{u}\right)\vec{u}+d\vec{N}\ \text{ and }\ \vec{N}=\frac{1}{d}\left\langle\vec{\ }-\left(\vec{\ },\vec{u}\right)\vec{u}\right\rangle \tag{49}$$

In view of the previous derived vector expression (48) , the expression for the acceleration of the second order $\vec{w}$ – jerk of the observed point $B$ of the rigid body is obtained, which rotates around a fixed point $O$ expressed in natural coordinates, it is possible separate it in three components as follows:

$$\vec{w}=\vec{w}_T+\vec{w}_N+\vec{w}_d \tag{50}$$

where the tangential $\vec{w}_T$, axial-normal $\vec{w}_N$ and complementary $+\vec{w}_d$ componentary-additional of the vector of the acceleration of the second order of an observed point $B$, of a rigid body rotating around a fixed point $O$, are expressed in natural coordinate system:

1* tangent component

$$\vec{w}_T=d\left(\ddot{\Omega}-\Omega\Omega^{2-}-\Omega^3\right)\vec{T}\qquad\qquad d=\left\|[\vec{u},\vec{\ }]\right\|=\overline{BB'} \tag{51}$$

2* normal (axipetal) component

$$\vec{w}_N=-3d\,\Omega\dot{\Omega}\vec{N} \tag{52}$$

3* complementary additional component

$$\begin{aligned}\vec{w}_d=&\left\langle\Omega\left(\dot{\vec{\Omega}},\vec{\ }\right)+2\,\Omega\left(\vec{\ },\left[\dot{\vec{\Omega}},\Omega\right]+2\Omega\left(\dot{\vec{\Omega}},\vec{\ }\right)\right)\right\rangle\vec{u}+\left\langle-2\Omega\left(\vec{\ },\vec{u}\right)\right\rangle\dot{\vec{\Omega}}-\\&-\left(\Omega,\vec{\ }\right)\ddot{\vec{\Omega}}+\left(\Omega,\dot{\vec{\Omega}}\right)\left[\vec{\Omega},\vec{\ }\right]+\left(\Omega,\vec{\ }\right)\left[\dot{\vec{\Omega}},\Omega\right]\end{aligned} \tag{53}$$

Since, in the general case, the rigid body's motion is around a fixed point $O$, and its instantaneous angular velocity $\Omega$ is equal to the sum of the angular velocities of *precession* $\dot{}$, around of the axis oriented by unit vector $\{\vec{k}\}$, *nutrition* $\dot{\vartheta}$, around of the axis oriented by unit vector $\{\vec{e}\}$, and *its own rotation* $\vartheta$, around of the axis oriented by unit vector $\{\vec{k}'\}$, we can write the following expression:

$$\{\Omega\}=\dot{\vartheta}\{\vec{e}\}+\dot{\ }\{\vec{k}\}+\vartheta\{\vec{k}'\}\qquad\text{or}\qquad \Omega=\dot{\vartheta}\vec{e}+\dot{\ }\vec{k}+\vartheta\vec{k}' \tag{54}$$

## 6. The natural coordinates of the vector of acceleration of the second order of a rigid body rotating around a fixed point in the case of regular progressive precession

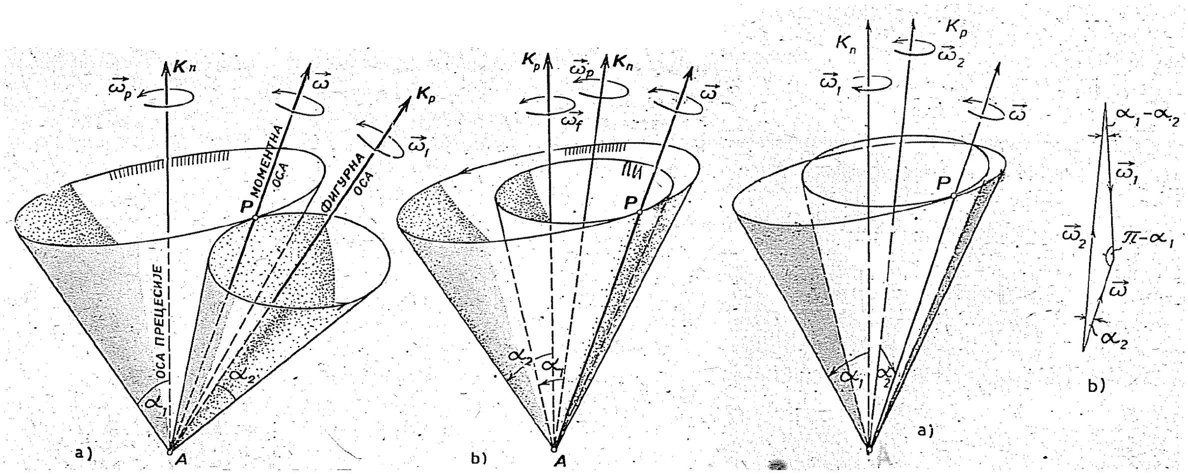
In the case where a rigid body rotates around a fixed point $O$ amd realizes by a regular progressive precession, *the angle of nutrition* is constant, $\vartheta=const$ , *and the angular velocity of the nutrient* is equal to zero, $\dot{\vartheta}=0$, and its angular velocity is:

$$\Omega=\dot{\ }\vec{k}+\vartheta\vec{k}'=\Omega\vec{k}+\Omega_s\vec{k}' \tag{55}$$

where: $\vartheta=const$ , $\dot{\vartheta}=0$ and *precession angular velocity* $\dot{\ }=\Omega$ and *self-rotation angular velocity* $\vartheta=\Omega_s$ around the figure axis.

In the case of a regular progressive precession of the body rotation around a fixed point $O$, from the vector expression (46) for angular acceleration, and it follows that an angular acceleration is of a constant value $\dot{\Omega}=0$, $\left|\frac{d\Omega}{dt}\right|=\left(\Omega\Omega_s\sin\vartheta_s\right)=const$ and always is in *the direction of the node axis,* oriented by a unit vector $\vec{e}$ :

$$\frac{d\vec{\Omega}_s}{dt} = \frac{d(\Omega_s \vec{u})}{dt} = \Omega_s[\vec{\Omega},\vec{u}] = [\vec{\Omega}, \vec{\Omega}_s] = = [\vec{\Omega}, \vec{\Omega}_s] = (\Omega\Omega_s \sin\vartheta_s)\vec{e} \tag{56}$$




a* b* c*

**Figure 5**. Axoils: stationary (herpolhoda) and movable (polhodia) of the rigid body rotation around fixed points in the case a* and b* regular progressive precession; c* angular acceleration of the retrograde precession of a rigid body rotating around fixed point (Figures a*, b* and c* from Reference [1])

In the case of a regular progressive precession, the jerk or the angular acceleration of the second order $\frac{d^2\vec{\Omega}}{dt^2}$, of the rigid body that rotates around the fixed point $O$, is also constant in value $\left|\frac{d^2\vec{\Omega}}{dt^2}\right| = (\Omega^2\Omega_s \sin\vartheta_s) = const$, obtained by expression (47), but it is in the direction of the transverse axis, oriented by the unit vector $\vec{c}$ :

$$\frac{d^2\vec{\Omega}}{dt^2} = (-\Omega^2)\vec{\Omega}_s + 2[\vec{\Omega}, \vec{\Omega}_s] + (\vec{\Omega}_s, \vec{\Omega})\vec{\Omega} = (\Omega^2\Omega_s \sin\vartheta_s)\vec{c} \tag{57}$$

because of:

$$\frac{d\vec{e}}{dt} = \dot{\varphi}\vec{c} = \vec{c}\Omega \tag{58}$$

Further, the velocity $\vec{v}$, acceleration $\vec{a}$ and acceleration of the second order (jerk) $\vec{w}$ of an observed point $B$, of the rigid body, which rotates around a fixed point $O$, in the case of regular progressive precession $\vec{\Omega} = \dot{\varphi}\vec{k} + \dot{\vartheta}\vec{k}' = \Omega\vec{k} + \Omega_s\vec{k}'$, are:

1* Velocity of an observed point $B$,

$$\vec{v} = \vec{\Omega} d\vec{T} = [\vec{\Omega}, \vec{\rho}] = [\vec{\Omega}_s + \vec{\Omega}, \vec{\rho}] \tag{59}$$

2* Components of the vector $\vec{a}$ of acceleration of an observed point $B$, tangential $a_T$, normal $a_N$ (normal-axipetal) and additional complementary $\vec{a}_d$, for the regular progressive precession are in the following forms:

$$a_T = 0 \qquad a_N = -\mathit{\Omega}^2 d\, \vec{N} \qquad \vec{a}_d = \left[\vec{\,}, \left[\vec{\Omega}, \vec{\mathit{\Omega}}\right]\right] = \left(\vec{\mathit{\Omega}}_s, \vec{\,}\right)\vec{\Omega} - \left(\vec{\Omega}, \vec{\,}\right)\vec{\mathit{\Omega}}_s \tag{60}$$

3* Components of the vector $\vec{w}$ of the acceleration of the second order of an observed point $B$, tangential $\vec{w}_T$, normal $\vec{w}_N$ (normal-axipetal) and additional complementary $\vec{w}_d$, for the regular progressive precession, are in the following forms:

$$\vec{w}_T = d\, \mathit{\Omega}\left(\Omega^2 + \mathit{\Omega}^2\right)\vec{T} \tag{61}$$

$$\vec{w}_N = 0 \tag{62}$$

$$\vec{w}_d = 2\left(\vec{\,}, \left[\vec{\Omega}, \vec{\mathit{\Omega}}\right]\right)\vec{\mathit{\Omega}} + \left(\vec{\mathit{\Omega}}, \vec{\Omega}\right)\left[\vec{\Omega}, \vec{\,}\right] + \left(\vec{\mathit{\Omega}}, \vec{\,}\right)\left[\vec{\Omega}, \vec{\mathit{\Omega}}\right] \tag{63}$$

**V. Concluding remarks**

In concluding remarks, we move from kinematics to the dynamics of rotation of the rigid body around a fixed point, using mass moment vectors with respect to a pole and the axis passing through this pole, introduced by K. (Stevanović) Hedrih 1992 (see References [9-14]), for description of rotation dynamics of a rigid body rotating around an axis.

Let us denote with $\vec{r}$ the position vector of a material point of the elementary mass $dm$ of the material rigid body in relation to the moment fixed point $O$ through which the instantaneous axis is oriented by the unit vector $\vec{n}$, which changes the direction, but always passes through one fixed point $O$, which we have chosen to be in the pole $O$. The axis around which the materially rigid body and its material point of elementary mass $dm$ are rotates, is the instantaneous axis of rotation, and of the body, both of the material points of the elementary mass $dm$.

Let around this instantaneous axis of current rotation, oriented by unit vector $\vec{n}$ which changes orientation, rotates, with angular velocity $\vec{\mathit{\Omega}} = \mathit{\Omega}\vec{n}$, that rigid body and this material point $dm$ of elemental mass, whose vector position $\vec{r}$ is determined at all times by its vector position $\vec{r}$ relative to the immovable pole at a fixed point $O$. The velocity $\vec{v}$ of the movement of this material point of the elementary mass $dm$ is equal to the vector product of the angular velocity $\vec{\mathit{\Omega}} = \mathit{\Omega}\vec{n}$ and its position vector $\vec{r}$: $\vec{v} = [\vec{\mathit{\Omega}}, \vec{r}] = \mathit{\Omega}[\vec{n}, \vec{r}]$. The velocity $\vec{v}$ of the rotational movement of the material point of the elementary mass $dm$ of the material heavy rigid body which rotates around the fixed point is governed by the instantaneous axis of the current rotation and its vector position, that is, on the vectors $\vec{n}$ and $\vec{r}$.

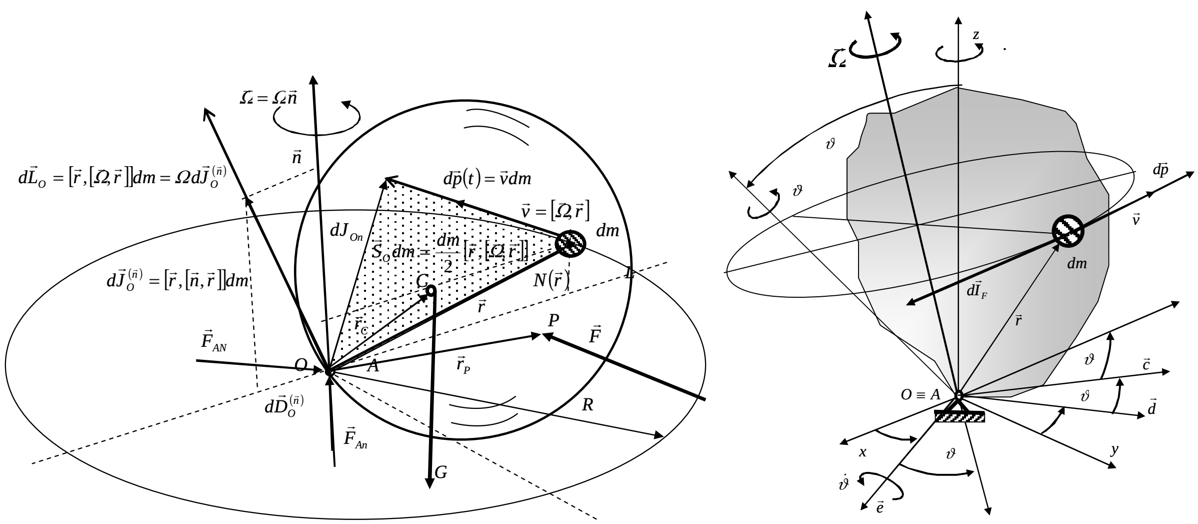
The vector of linear momentum (of impulse of motion) of the material mass particle $d\vec{p}(t)$, which rotates at an angular velocity $\vec{\mathit{\Omega}} = \mathit{\Omega}\vec{n}$, around the momentary axis of the unit vector $\vec{n}$, which passes through the moment fixed point $O$, is:

$$d\vec{p}(t)=\vec{v}dm=[\vec{\Omega},\vec{r}]dm=\Omega[\vec{n},\vec{r}]dm=\Omega d\vec{S}_O^{(\vec{n})} \tag{64}$$

where notation is introduced for

$$d\vec{S}_O^{(\vec{n})}\overset{def}{=}[\vec{n},\vec{r}]dm \tag{65}$$

and vector definition for the vector $d\vec{S}_O^{(\vec{n})}$ of the static moment of the mass of the material point of the elementary mass $dm$ relative to the fixed point $O$ and the axis oriented by the unit vector $\vec{n}$, which passes through this fixed point $O$. It is also the mass moment of the first order or the linear moment of the mass of the material point of the elementary mass $dm$ in relation to the moment point $O$ and the axis oriented by the unit vector $\vec{n}$, which passes through this fixed point $O$.




a*  b*

**Figure 6.** a* Graphical presentation of the vector kinetic parameters of a rigid body and its elementary mass particle which rotates around a fixed point: Vevtors of linear and angular momentum; b* Euler's angles and component's angular velocities

A vector $\vec{p}(t)$ of the impulse of the motion of the system of material particles of an elementary mass $dm$, which consists of a rigid material body that rotates around a fixed point $O$, which can be represented as a rotation around the instantaneous axis of the current rotation oriented by the unit vector $\vec{n}$, is written by the vector sum of the vector of impulses of motion of all material particles of elementary masses $dm$, which are elementary parts of a rigid material body, rotating at an angular velocity $\vec{\Omega}=\Omega\vec{n}$, around an axis oriented by a unit vector $\vec{n}$, which passes through the momentary, immobile (fixed) point $O$. This vector sum is integral to the volume of the body:

$$\vec{p}(t)=\iiint_V \vec{v}dm=\iiint_V[\vec{\Omega},\vec{r}]dm=\Omega\iiint_V[\vec{n},\vec{r}]dm=\Omega\vec{S}_O^{(\vec{n})} \tag{66}$$

where notation is introduced for

$$\vec{S}_O^{(\vec{n})} = \iiint_V [\vec{n}, \vec{r}] dm \tag{67}$$

and the vector definition for the static moment vector $\vec{S}_O^{(\vec{n})}$ of the mass of a rigid material body.

An angular momentum vector $\vec{L}_O$ of the motion of a system of material particles of an elementary mass $dm$ that makes up a solid material body that rotates around a fixed point $O$, that can be represented as a rotation around the instantaneous axis of rotation oriented by a vector sum of moment of momentum vectors of all material particles of elementary masses, which are elementary parts of a rigid material body rotating at an angular velocity $\vec{\Omega} = \Omega \vec{n}$ around an instantaneous axis oriented by a unit vector $\vec{n}$ that passes through a momentary point is:

$$\vec{L}_O = \iiint_V d\vec{L}_O \overset{def}{=} \iiint_V [r, d\vec{p}] = \iiint_V [r, \vec{v}] dm = \iiint_V [\vec{r}, [\vec{\Omega}, \vec{r}]] dm = \Omega \iiint_V [\vec{r}, [\vec{n}, \vec{r}]] dm = \Omega \vec{J}_O^{(\vec{n})} \tag{68}$$

where notation is introduced for

$$\vec{J}_O^{(\vec{n})} \overset{def}{=} \iiint_V [\vec{r}, [\vec{n}, \vec{r}]] dm \tag{69}$$

and vector definition for the mass moment vector $\vec{J}_O^{(\vec{n})}$ of the inertia of the mass of the material body relative to the moment fixed point $O$ and the axis oriented by the unit vector $\vec{n}$, which passes through this fixed point $O$. This is the vector $\vec{J}_O^{(\vec{n})}$ of the moment of the mass of the second order or the square moment of the mass of the material body in relation to the fixed point $O$ and the axis orientated by the unit vector $\vec{n}$, which passes through this fixed point (for details see References [8-14]).

The vector $\vec{J}_O^{(\vec{n})}$ of the moment of inertia of the mass of a rigid material body in relation to the fixed point $O$ and the instantaneous axis oriented by a unit vector $\vec{n}$, which passes through this fixed point $O$, can be explained in two components, one $J_{On}$ in the direction of the axis $\vec{n}$ and the other $\vec{\mathfrak{D}}_O^{(\vec{n})}$ orthogonal to that instantaneous axis $\vec{n}$, and both in the deviation plane. The vectors $\vec{J}_O^{(\vec{n})}$, $\vec{n}$ and $\vec{\mathfrak{D}}_O^{(\vec{n})}$, we can write in the following form:

$$\vec{J}_O^{(\vec{n})} = \left(\vec{n}, \vec{J}_O^{(\vec{n})}\right)\vec{n} + \left[\vec{n}, \left[\vec{J}_O^{(\vec{n})}, \vec{n}\right]\right] = J_{On}\vec{n} + \vec{\mathfrak{D}}_O^{(\vec{n})} \tag{61}$$

By using d'Alambert principle of the dynamical equilibrium or theorems of the change in time of the linear and angular momentum, two vector equations can be written, taking into account external forces and reaction applied to the rigid body in rotation motion around a fixed point, in the following forms [5, 6]:

$$\frac{d\vec{p}(t)}{dt} = \left|\vec{S}_O^{(\vec{n})}\right| \vec{\mathfrak{R}}_1 = \vec{F}_{AN} + \vec{F}_{An} + \vec{F}_B + \vec{F} + \vec{G} \tag{62}$$

$$\frac{d\vec{L}_O}{dt} = \Omega\left(\vec{J}_O^{(\vec{n})}, \vec{n}\right)\vec{n} + \left|\vec{\mathfrak{D}}_O^{(\vec{n})}\right| \vec{\mathfrak{R}} = \left[\vec{r}_P, \vec{F}\right] + \left[\vec{r}_C, \vec{G}\right] + \left[\vec{r}_B, \vec{F}_B\right] \tag{63}$$

or in the form

$$\frac{d\vec{p}(t)}{dt} = \dot{\Omega}\vec{\mathfrak{S}}_O^{(\vec{n})} + \Omega\dot{\vec{\mathfrak{S}}}_O^{(\vec{n})} + \Omega\left[\vec{\Omega}, \vec{S}_O^{(\vec{n})}\right] = \sum_{k=1}^{S} \vec{F}_k + \vec{G} + \vec{F}_{AN} + \vec{F}_{An} \tag{64}$$

$$\frac{d\vec{L}_O}{dt} = \dot{\Omega}\vec{J}_O^{(\vec{n})} + \Omega\overset{*}{\vec{J}}_O^{(\vec{n})} + \Omega\left[\vec{\Omega}, \vec{J}_O^{(\vec{n})}\right] = \sum_{k=1}^{S}\left[\vec{r}_k, \vec{F}_k\right] + \left[\vec{r}_C, \vec{G}\right] \tag{65}$$

or

$$\frac{d\vec{L}_O}{dt} = \overset{*}{\vec{L}}_O + \left[\vec{\Omega}, \vec{L}_O\right] = \sum_{k=1}^{S}\left[\vec{r}_k, \vec{F}_k\right] + \left[\vec{r}_C, \vec{G}\right] = \vec{\mathfrak{M}}_0 \tag{66}$$

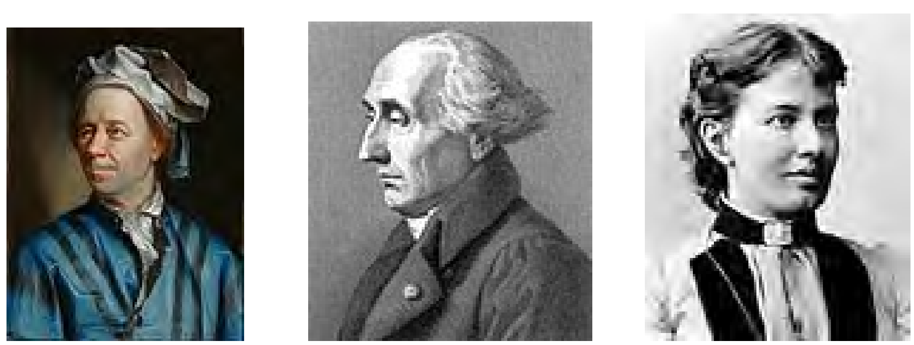
Taking that is $\vec{L}_O = \Omega\vec{J}_O^{(\vec{i}')} + \Omega\vec{J}_O^{(\vec{j}')} + \Omega\vec{J}_O^{(\vec{k}')}$ the following scalar equations, known in literature can be obtained:

$$\begin{aligned} \dot{L}_O + \Omega L_O - \Omega L_O &= \mathfrak{M}_O \\ \dot{L}_O + \Omega L_O - \Omega L_O &= \mathfrak{M}_O \\ \dot{L}_O + \Omega L_O - \Omega L_O &= \mathfrak{M}_O \end{aligned} \tag{67}$$

These equations are the Euler's differential equations of the dynamics of a rigid material body that rotates around a fixed point [2].

$$\begin{aligned} \dot{\Omega}_1 J_{O1} - \Omega_2\Omega_3\left(J_{O2} - J_{O3}\right) &= \mathfrak{M}_{O1} \\ \dot{\Omega}_2 J_{O2} - \Omega_1\Omega_3\left(J_{O2} - J_{O1}\right) &= \mathfrak{M}_{O2} \\ \dot{\Omega}_3 J_{O3} - \Omega_1\Omega_2\left(J_{O1} - J_{O2}\right) &= \mathfrak{M}_{O3} \end{aligned} \tag{68}$$

Problem is to solving previous three Euler's differential equations of the dynamics of a rigid material body taking into account three kinematical equations containing three Euler's angles. But up to these days, this problem hasn't been solved in general case. Some special cases were presented by Euler, Lagrange and Kovalevskaya:



a* b* c*

**Figure 7. a*Leonhard Paul Euler** (Basel, 15 April 1707 – Sankt Peterburg, 18 September 1783), b* **Joseph-Louis Lagrange** ( born **Giuseppe Luigi Lagrangia** or **Giuseppe Ludovico De la Grange Tournier**, 25 January 1736 – 10 April 1813), and c* **Sofia Vasilyevna Kovalevskaya** (Russian: Со́фья Васи́льевна Ковале́вская), born **Sofia Vasilyevna Korvin-Krukovskaya** (15 January / 3 January] 1850 – 10 February 1891))

1 * **Euler's solution:** Euler's solution refers to a rigid heavy body with a fixed point, around which the rigid body rotates, at the center of mass. This means that the rotation is without action of the gravitational force, because the resultant of the force of gravity passes through a fixed point (center of mass) of the body and affects only the hinge reaction and its moment at the fixed point is equal to zero.


Figure 8. **Leonhard Paul Euler** (Basel, 15 April 1707 – Sankt Peterburg, 18 September 1783)

2 * **Lagrange’s solution**: Lagrange's solution refers to an axially symmetric rigid body with the center of mass on it, and with a fixed point around which the rigid body faces the axis of symmetry.

3 * **Kowalewskaya solution:** The Kowalewskaya solution relies on a rigid heavy body with two main axial mass moment inertia for the main axes through the center of mass equal, and the third major is twice as big as them. A center of mass, which lies at the equatorial plane of the frieze of the main axes of the mass inertia moments for which the axial moments of mass inertia are equal, and which passes through the fixed point around which the rigid heavy body turns.

In 1889 Sophiya **Kowalewskaya** solved two problems. The first problem is: Find all rigid bodies, which rotate around a fixed point, in the presence of gravity, such that the system of differential equations of motion is integrable in the sense of Kovalevskaya. This means that the system has solutions, which can be expressed as Lauren's order in the function of time, containing a set of free parameters equal to the number of degrees of freedom minus one; the dimensional phase space for such a rigid body is. Strict conditions of integrability gave the following already known solutions.

There are many attempts and the majority is reduced to numerical approximations for real bodies in a three-dimensional space.

**Acknowledgement.** Parts of this research were supported by the Ministry of Sciences of Republic Serbia trough Mathematical Institute SANU Belgrade Grants OI 174001” Dynamics of hybrid systems with complex structures”. Also, author is very grateful to young researcher and Ph. D student Stepa Paunović, for valuable help in improvement in English of the manuscript,

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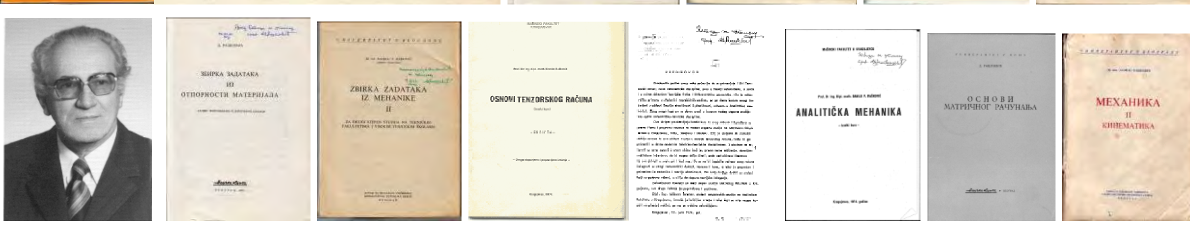
# APPENDIX 1.:


Presentation of a collections with university books some as monograph, for Yugoslav and Serbian students, written by Profess and ScDr. Ing Danolo P. Rašković (September 10,

(August 28) 1910 in Užice-January 29, 1985 in Belgrade). Prof. Dr. Ing. Dipl. Math. Danilo P. Rašković was the first head of the Department of Mechanics and Automatics within the Faculty of Mechanical Engineering of the University of NiŠ. This distinguished scientific figure of exquisite creative energy and inspired enthusiasm, a scholar deeply attached to the Yugoslav and Serbian scientific and cultural heritage and an exquisite pedagogist of high ethic principles is in the living memory of many generations of students whom he taught how to learn and love mechanics, as a basic scientific branch of mechanical engineering either directly, through his lectures, or through his various and numerous textbooks and collections of problems which circulate in more than 140.000 copies.
Prof. Dr. Ing. Dipl. Math. Danilo P. Rašković lectured mechanics, strength of materials and oscillation theory at the faculties of mechanical engineering in Belgrade, Niš, Kragujevac, Novi Sad and Mostar, as well as in the Faculty of Science in Belgrade, Faculty of Philosophy in Novi Sad, Faculty of Electronics in Niš and at the Military-Technical College in Belgrade.
First and unique book in Theory of Oscillations (editions: 1957, 1965 – in 6.000 copies) written in Serbian language is authored by Danilo P. Rašković in 1957 and in the form of monograph. Last published books, authored by Danilo P. Rašković is Theory of elasticity (1985 edition, 2,000 copies. ) , Analytical Mechanics and Tensor Calculus.

# APPENDIX 2. :

## The Round Table discussion at **10th International Conference STABILITY, CONTROL AND RIGID BODIES DYNAMICS, Donetsk (Ukraine), June 5-10, 2008.**, organized by academician NANU Alexander Kovalev.

http://www.iamm.ac.donetsk.ua/icscd.html

ICSCD 08

Institute of Applied Mathematics and Mechanics of National Academy of Sciences of Ukraine (IAMM NASU) together with Donetsk National University (DonNU) organizes the 10th International Conference "Stability, Control and Rigid Bodies Dynamics" in Donetsk in June, 2008. Chairman of the Organizing Committee of the Conference is Alexander M. Kovalev (IAMM NASU.

Not long ago D.L.Abrarov (Russia) has published two books and V.N.Onikiychuk (Russia) has published one book containing the authors results that give, by their opinion, the exact solution of the problem on motion of a rigid body with a fixed point in the field of gravity. This problem is one of the classical problems of mechanics and it was one of the main topics of the conference series "Stability, control and rigid bodies dynamics". Therefore the scientific community present at the 10th Conference had to express its attitude to these publications.

For this purpose the Round Table discussion was held on the theme "Integrability problem for equations of rigid body dynamics". It was organized in the following way: at first

D.L.Abrarov and V.N.Onikiychuk told about their results, and then the consideration of the theme took place. Following persons participated in this consideration: I.N.Gashenenko (Ukraine), A.A.Ilyukhin (Russia), M.P.Kharlamov (Russia), A.M.Kovalev (Russia), T.S.Krasnopolskaya (Ukraine), M.E.Lesina (Ukraine), V.V.Meleshko (Ukraine), V.V.Sokolov (Russia), V.N.Tkhai (Russia), H.M.Yehia (Egypt), Katica (Stevanović) Hedrih (Serbia) and others.

They discussed the works of D.L.Abrarov on the exact solvability of the Euler-Poisson equations in terms of exponents of $L$-functions of elliptic curves over the field of rational numbers. Their opinions can be summarized in the following way. In his works, D.L.Abrarov makes an attempt to give a new mathematical description of the problem. It is necessary to interpret the obtained results in the Language which is accepted among the specialists in the given scientific area and to present the strictly grounded facts that can be verified.

The abstract of V.N.Onikiychuk's communication was not published in the Book of abstracts, the Organizing Committee decided to give him an opportunity to speak at the Round Table. The participants of the conference suggested that the author should state his results in the scientific articles for submitting to the specialized journals, with attention to the opinion of reviewers and authoritative scholars.

dynamics and celestial mechanics, 10th International Conference STABILITY, CONTROL AND RIGID BODIES DYNAMICS, Donetsk (Ukraine), June 5-10, 2008. http://www.iamm.ac.donetsk.ua/icscd.html

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# APPENDIX 1

Presentation of a collections with university books some as monograph, for Yugoslav and Serbian students, written by Profess and ScDr. Ing Danilo P. Rašković (September 10, (August 28) 1910 in Užice-January 29, 1985 in Belgrade). Prof. Dr. Ing. Dipl. Math. Danilo P. Rašković was the first head of the Department of Mechanics and Automatics within the Faculty of Mechanical Engineering of the University of Niš. This distinguished scientific figure of exquisite creative energy and inspired enthusiasm, a scholar deeply attached to the Yugoslav and Serbian scientific and cultural heritage and an exquisite pedagogist of high ethic principles is in the living memory of many generations of students whom he taught how to learn and love mechanics, as a basic scientific branch of mechanical engineering either directly, through his lectures, or through his various and numerous textbooks and collections of problems which circulate in more than 140.000 copies.


Prof. Dr. Ing. Dipl. Math. Danilo P. Rašković lectured mechanics, strength of materials and oscillation theory at the faculties of mechanical engineering in Belgrade, Niš, Kragujevac, Novi Sad and Mostar, as well as in the Faculty of Science in Belgrade, Faculty of Philosophy in Novi Sad, Faculty of Electronics in Niš and at the Military-Technical College in Belgrade.

First and unique book in Theory of Oscillations (editions: 1957, 1965 – in 6.000 copies) written in Serbian language is authored by Danilo P. Rašković in 1957 and in the form of monograph. Last published books, authored by Danilo P. Rašković is Theory of elasticity (1985 edition, 2,000 copies. ) , Analytical Mechanics and Tensor Calculus.

# APPENDIX 2

## The Round Table discussion at 10th International Conference

**STABILITY, CONTROL AND RIGID BODIES DYNAMICS, Donetsk (Ukraine), June 5-10, 2008.**, organized by academician NANU Alexander Kovalev.
http://www.iamm.ac.donetsk.ua/icscd.html

ICSCD 08

Institute of Applied Mathematics and Mechanics of National Academy of Sciences of Ukraine (IAMM NASU) together with Donetsk National University (DonNU) organizes the 10th International Conference "Stability, Control and Rigid Bodies Dynamics" in Donetsk in June, 2008. Chairman of the Organizing Committee of the Conference is Alexander M. Kovalev (IAMM NASU.

Not long ago D.L.Abrarov (Russia) has published two books and V.N.Onikiychuk (Russia) has published one book containing the authors results that give, by their opinion, the exact solution of the problem on motion of a rigid body with a fixed point in the field of gravity. This problem is one of the classical problems of mechanics and it was one of the main topics of the conference series "Stability, control and rigid bodies dynamics". Therefore the scientific community present at the 10th Conference had to express its attitude to these publications.

For this purpose the Round Table discussion was held on the theme "Integrability problem for equations of rigid body dynamics". It was organized in the following way: at first D.L.Abrarov and V.N.Onikiychuk told about their results, and then the consideration of the theme took place. Following persons participated in this consideration: I.N.Gashenenko (Ukraine), A.A.Ilyukhin (Russia), M.P.Kharlamov (Russia), A.M.Kovalev (Russia), T.S.Krasnopolskaya (Ukraine), M.E.Lesina (Ukraine), V.V.Meleshko (Ukraine), V.V.Sokolov (Russia), V.N.Tkhai (Russia), H.M.Yehia (Egypt), Katica (Stevanović) Hedrih (Serbia) and others.

They discussed the works of D.L.Abrarov on the exact solvability of the Euler-Poisson equations in terms of exponents of *L*-functions of elliptic curves over the field of rational numbers. Their opinions can be summarized in the following way. In his works, D.L.Abrarov makes an attempt to give a new mathematical description of the problem. It is necessary to interpret the obtained results in the Language which is accepted among the specialists in the given scientific area and to present the strictly grounded facts that can be verified.

The abstract of V.N.Onikiychuk's communication was not published in the Book of abstracts, the Organizing Committee decided to give him an opportunity to speak at the Round Table. The participants of the conference suggested that the author should state his results in the scientific articles for submitting to the specialized journals, with attention to the opinion of reviewers and authoritative scholars.

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ISBN 978-83-7051-956-8

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Siedlce, April 05, 2019

**Professor Dr**
**Katica R. (Stevanović) Hedrih**
**Department of Mechanics**
**Mathematical Institute SANU**
**Knez Mihailova st. 36/III**
**Belgrade, 11009 Serbia**
**khedrih@sbb.rs**

Dear Professor Dr Katica (Stevanović) Hedrih,

On behalf of the Organizing Committee of the 10th International Workshop on Computer Algebra Systems in Teaching and Research (CASTR'2019) to be held in Siedlce, Poland from September 25 till September 29, 2019, we are glad to invite you to participate in the conference and to present your talks:

1. Acceleration of the second order (Jerk) of a kinetic point moving with constant sectorial velocity as well as of the planet moving
2. Acceleration of the second order – jerk of a rigid body rotates around a fixed point

As an invited speaker you'll have 45 min. for presentation. We also expect that you'll agree to act as a Chairman at the mini-symposium on Classical and Celestial Mechanics.

Please note that the registration fee is 500 PLN. It covers organization expenses, conference materials and refreshment room at the Workshop. It can be paid on site upon arrival. Participants of the workshop CASTR'2019 will be accommodated at the UPH hotel situated in a walking distance from the Faculty of Science building, Zytnia str. 17/19, Siedlce, (http://www.uph.edu.pl/sprawy-studenckie/domy-studenckie).

We kindly ask you to confirm your participation by September 15, 2019 via e-mail: castr@uph.edu.pl and to let us know some details concerning your arrival and departure (number of train or flight, date and time). All the details concerning the workshop you can find on the website http://www.castr.uph.edu.pl

We look forward to seeing you in Siedlce.

Best wishes,

PROREKTOR DS. NAUKI

dr Mirosław Minkina
profesor nadzwyczajny

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Siedlce, September 25-29, 2019

2019 C@STR
Computer Algebra Systems in Teaching and Research

# Certificate of Attendance

This certificate is presented to

## Katica R. Hedrih

**Department of Mechanics,**
**Mathematical Institute of the Serbian Academy of Science and Arts,**
**Belgrade, Serbia**

For attendance at

CASTR'2019, 10th International Workshop on Computer Algebra Systems in Teaching and Research (www.castr.uph.edu.pl)

Held in Siedlce, Poland, September 25-29, 2019.

Dr. Marek Siłuszyk

Organizing Committee of CASTR'2019
University of Natural Sciences and Humanities in Siedlce
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Tel. +48 25 6431076
http://www.castr.uph.edu.pl

CASTR' 2019, September 26-29, Siedlce, Poland

Siedlce, Poland, Sepember 25 – 29, 2019

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- Modelling of the Galactic Cosmic Rays Propagation
- Computer methods in Classical and Celestial Mechanics
- Mathematical Modelling in Physics, Finance and Economics
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2019 C@STR

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**Mini-symposium on Computational Aspects of Classical and Celestial Mechanics, Stability and Motion Control is planned to be organized in the framework of the workshop on September 27, 2019.**

**Organizers:**
Pavel S. Krasilnikov (Moscow, Russia),
Katica R. (Stevanovic) Hedrih (Belgrade, Serbia),
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